Arithmetic holonomy bounds

(G-functions in shallow waters)

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A classical theorem

Theorem (Raphaël Robinson, 1960s)

$$\mathcal{O}(\mathbf{C} \smallsetminus [1,\infty)) \cap \mathbf{Z}[\![x]\!] = \mathbf{Z}[x, 1/(1-x)]$$

$$\mathcal{O}\left(\mathbf{C} \smallsetminus [\phi^{-1},\infty)\right) \cap \mathbf{Z}[\![x]\!] = \mathbf{Z}[x, 1/(1-x), 1/(1-x-x^2)]$$

$$\#\mathcal{O}(\mathbf{C} \smallsetminus [1/4,\infty)) \cap \mathbf{Z}[\![x]\!] = \#\mathbf{R}.$$

$\mathcal{O}(\widetilde{\mathcal{V}})$, where $\widetilde{\mathcal{V}} = \{\mathrm{Spf}\, \mathbf{Z}[\![x]\!] \text{ glued to } x \in \mathbf{C} \smallsetminus [r,\infty)\}$ is an instance of Bost's *formal-analytic arithmetic surface*

$$\frac{1}{x^n \cdot 2T_n\left(\frac{1}{2x} - 1\right)} \in \mathcal{O}(\mathbf{C} \smallsetminus [1/(2+2\cos(\pi/(2n))),\infty)) \cap \mathbf{Z}[\![x]\!]$$

$$\subset \mathcal{O}(\mathbf{C} \smallsetminus (1/4,\infty)) \cap \mathbf{Z}[\![x]\!].$$

$$\frac{1+\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n \in \mathbf{Z}[\![x]\!], \quad \det\left(C_{i+j}\right)_{i,j=0}^{k} \equiv 1.$$

A proof quintessence, rational version

Rational version: Pólya observed the use of the Hankel determinants $\det(a_{i+j})_{i,j=0}^{n}$: they are $\leq \mathrm{cap}_{[\infty]}(\mathcal{K})^{n^2+o(n^2)}$ if $\sum_{n=0}^{\infty} a_n/z^n \in \mathbf{C}[\![1/z]\!]$ is holomorphic on $\widehat{\mathbf{C}} \smallsetminus \mathcal{K}$.

> $\exists$ *continuous* Green's function $g_{\mathcal{K}} : \mathbf{C} \to \mathbf{R}$ *which is zero on* $\mathcal{K}$, *harmonic on* $\mathbf{C} \smallsetminus \mathcal{K}$, *and asymptotically* $g_{\mathcal{K}}(z) = \log\left|z/\mathrm{cap}_{[\infty]}(\mathcal{K})\right| + o_{|z|\to\infty}(1)$ *near* $z = \infty$.

Zudilin (*A determinantal approach to irrationality*) analyzed the situation of denominators: if $A^{n+1}[1,\ldots,bn]^{\sigma} a_n \in \mathbf{Z}$, then as soon as

$$\mathrm{cap}_{[\infty]}(\mathcal{K}) < \left(Ae^{b\sigma}\right)^{-3/2},$$

we still have the identical vanishing $\det(a_{i+j})_{i,j=0}^{n} \equiv 0$ for all $n \geq n_0$, which — a theorem of Kronecker — is tantamount to $f(x) \in \mathbf{C}(x)$.

A proof quintessence, integral version

Let us limit to the example of $\mathcal{O}(\widetilde{\mathcal{V}}) = \mathbf{Z}[x, 1/(1-x)]$ for $r > \phi^{-1}$.

Find a magical polynomial, in this case $h(x) := x^2 - x \in \mathbf{Z}[x] \smallsetminus \mathbf{Z}$, with $\sup_{[0,r^{-1}]} |h| < 1$. The purpose is to prove that $h(x)^m f(1/x) \in \mathbf{Z}[x]$ (a polynomial) for some $m \geq m_0$.

We have for the first negative degree coefficient:

$$[x^{-2k}]\,\{h(x)^m f(1/x)\} = \frac{1}{2\pi i} \oint_\Gamma h(z)^{m+k} f(1/z)\, \frac{dz}{z},$$
$$[x^{-2k-1}]\,\{h(x)^m f(1/x)\} = \frac{1}{2\pi i} \oint_\Gamma h(z)^{m+k} f(1/z)\, dz,$$

with a small integrand if the integration contour $\Gamma \subset \mathbf{C} \smallsetminus [r, \infty)$ is chosen to traverse the slit sufficiently nearly and far out.

Holonomic functions

Definition.
We denote by $\mathrm{Hol}_{\{a_1,\ldots,a_k\}}(\mathbf{P}^1)$ *the* $\mathbf{C}$*-algebra of formal power series* $f(x) \in \mathbf{C}[\![x]\!]$ *that continue analytically as holomorphic functions along all paths in* $\mathbf{P}^1 \smallsetminus \{a_1, \ldots, a_k\}$*.*

("Extend as functions on the universal cover of $\mathbf{P}^1 \smallsetminus \{a_1, \ldots, a_k\}$")

Example. This is the case if $f(x)$ satisfies a linear ODE $\mathcal{L}(f) = 0$ for some nonzero linear differential operator $\mathcal{L}$ over $\mathbf{P}^1$ without singularities outside $\{a_1, \ldots, a_k\}$.

A recent theorem

Theorem (CDT § 2.7 and § 2.8)

As modules over the respective $\mathcal{O}(\widetilde{\mathcal{V}})$*:*

(a)

$$\mathcal{O}\left(\mathbf{C} \smallsetminus [1,\infty)\right) \cap \bigoplus_{n=0}^{\infty} \frac{x^n}{[1,\ldots,n]}\,\mathbf{Z}$$
$$= \mathrm{Span}_{\mathbf{Z}[x,1/(1-x)]}\left\{\log(1-x);\ 1;\ x^q/q : \text{prime powers } q\right\}.$$

(b)

$$\mathrm{Hol}_{\{0,1,\infty\}}(\mathbf{P}^1) \cap \bigoplus_{n=0}^{\infty} \frac{x^n}{[1,\ldots,n][1,\ldots,n/2]}\,\mathbf{Z}$$
$$= \mathrm{Span}_{\mathbf{Z}[x,1/(1-x)]}\Big\{\log(1-x), \log^2(1-x);\ 1;$$
$$x^{q_1+q_2}/q_1 q_2 : \text{prime powers } q_1, q_2\Big\}.$$

An unsolved problem

Problem

Compute the $\mathbf{Z}[x, 1/(1-x)]$*-module*

$$\mathrm{Hol}_{\{0,1,\infty\}}(\mathbf{P}^1) \cap \bigoplus_{n=0}^{\infty} \frac{x^n}{[1,\ldots,n]^2}\,\mathbf{Z}$$

Is it finite over $\mathbf{Q}[x, 1/(1-x)]$*?*

Theorem

It is finite over $\mathbf{Q}(x)$*, with — unknown — dimension in* $\{5,6,7,8,9\}$*, and containing at least the elements*

$$1, \quad \log(1-x), \quad \log^2(1-x), \quad \mathrm{Li}_2(x), \quad \frac{1}{\sqrt{1-x}}\int_0^x \frac{\log(1-t)\,dt}{t\sqrt{1-t}}$$

G-functions

Notation: $\mathscr{D} \subset \mathbf{C}[\![x]\!]$ to denote the ring of D-finite power series: $\mathcal{L}(f) = 0$ for some nonzero linear differential operator $\mathcal{L}$ with $\mathbf{C}[x]$ coefficients

G-functions (may take $\{a_1, \ldots, a_k\}$ the singularities of $\mathcal{L}$).

$$f(x) \in \mathscr{D} \cap \mathrm{Hol}_{\{a_1,\ldots,a_k\}}(\mathbf{P}^1) \cap \bigoplus_{n=0}^{\infty} \frac{x^n}{A^{n+1}[1,\ldots,bn]^{\sigma}}\,\mathbf{Z}.$$

Conjecturally — and certainly for the G-functions of geometric origin — the denominator type will be always of this form for some $A \in \mathbb{N}^{>0}$, $b \in \mathbf{Q}^{\geq 0}$, and $\sigma \in \mathbb{N}$.

More precisely: if $\mathcal{L}(f) = 0$ is a minimal order linear ODE of f, then one can take $\sigma = \mathrm{rank}(\mathcal{L}) - 1$ and b to be any real number exceeding the LCM of the $x = 0$ local exponents of $\mathcal{L}$.

An open problem

For which $\sigma \in \mathbb{N}$ *does*

$$\mathscr{D} \cap \mathrm{Hol}_{\{0,1,\infty\}}(\mathbf{P}^1) \cap \bigoplus_{n=0}^{\infty} \frac{x^n}{[1,\ldots,n]^\sigma}\,\mathbf{Z}$$

generate a finite $\mathbf{Q}(x)$*-module?*

Example: the multiple polylogarithms ring in one variable

The $\mathbf{Z}[x, 1/(1-x)]$-algebra $\mathrm{MP} \subset \mathbf{Q}[\![x]\!]$ generated by the functions

$$\mathrm{Li}_{k_1,\ldots,k_d}(x) = \sum_{n_1>n_2>\ldots>n_d} \frac{x^{n_1}}{n_1^{k_1} n_2^{k_2} \cdots n_d^{k_d}}$$

meets the triple intersection conditions with floating σ, but gives only finitely many $\mathbf{Q}(x)$-linear independent elements below a given σ.

More elements

$$J_k := \frac{1}{\sqrt{1-x}} \int_0^x \frac{\log^{2k-1}(1-x)\,dt}{t\sqrt{1-t}}, \qquad k = 1, 2, 3, \ldots$$

$$\in \mathrm{Hol}_{\{0,1,\infty\}}(\mathbf{P}^1) \cap \bigoplus_{n=0}^{\infty} \frac{x^n}{[1, \ldots, n]^{2k}} \mathbf{Z}$$

Adjoin these, plus all their dx/x and $dx/(1-x)$ iterated integrals, to the multiple polylogarithms ring.
Any other other (= $\mathbf{Q}(x)$-linearly independent) functions in

$$\mathscr{D} \cap \mathrm{Hol}_{\{0,1,\infty\}}(\mathbf{P}^1) \cap \bigoplus_{n=0}^{\infty} \frac{x^n}{[1, \ldots, n]^{\bullet}} \mathbf{Z}?$$

G-functions

David and Gregory Chudnovsky's fundamental theorem (1984)

An irreducible linear homogeneous ODE $\mathcal{L}(f) = 0$ *over* $\mathbf{Q}(x)$ *that admits one nonzero* G*-series solution* $f(x) \in \overline{\mathbf{Q}}[\![x]\!]$ *at* $x = 0$ *admits in fact a full set of* $\mathbf{C}$*-linearly independent* G*-series solutions*

$$g_1(x-\alpha), \ldots, g_{\mathrm{rank}(\mathcal{L})}(x-\alpha) \in \overline{\mathbf{Q}}[\![x-\alpha]\!]$$

at every nonsingular point $x = \alpha \in \overline{\mathbf{Q}}$.

References: Dwork–Gerotto–Sullivan; Beukers's *E-functions and G-functions* (Notes from the 2008 Arizona Winter School); Gabriel Lepetit's *Le Théorème d'André–Chudnovsky–Katz* (2021)

Using this, most of the fundamental questions — including the Bombieri–Dwork, Grothendieck–Katz, or the above denominators structure conjecture (Fischler–Rivoal) — all reduce to the case $\{a_1, \ldots, a_k\} = \{0, 1, \infty\}$:

$$u_*(\mathcal{E}, \nabla) \quad \text{under an étale covering } u : U \to \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$$

$$\{a_1, \ldots, a_k\} \rightsquigarrow \{0, 1, \infty\}$$

G-functions in shallow waters

Main object: the ring

$$\mathcal{R}_{A,b,\sigma} := \mathrm{Hol}_{\{0,1,\infty\}}(\mathbf{P}^1) \cap \bigoplus_{n=0}^{\infty} \frac{x^n}{A^{n+1}[1,\ldots,bn]^{\sigma}} \mathbf{Z}$$

There is a universal holomorphic map $\lambda : \mathbf{D} \to \mathbf{C} \setminus \{1\}$ subject to $\lambda^{-1}(0) = \{0\}$:

$$\lambda(q) := \frac{\left(\sum_{n \in 1+2\mathbf{Z}} q^{n^2/4}\right)^4}{\left(\sum_{n \in 2\mathbf{Z}} q^{n^2/4}\right)^4} = 16q \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^8, \quad q := e^{\pi i \tau}.$$

Then $\lambda^* \mathcal{R} \hookrightarrow \mathcal{O}(\mathbf{D})$, but also $\mathbf{Z}[\![q]\!] = \mathbf{Z}[\![\lambda/16]\!]$.

G-functions in shallow waters

$$\mathcal{R}_{A,b,\sigma} := \left\{ f(x) \in \bigoplus_{n=0}^{\infty} \frac{x^n}{A^{n+1}[1,\ldots,bn]^{\sigma}} \mathbf{Z} \,:\, f(\lambda(z)) \in \mathcal{O}(\mathbf{D}) \right\}$$

Some things could be proved:

- $\mathcal{R}_{1,0,0} = \mathbf{Z}[x, 1/(1-x)]$
- $\mathcal{R}_{A,b,\sigma}$ generates a finite-dimensional $\mathbf{Q}(x)$-vector space if and only if $Ae^{b\sigma} < 16$
- In particular, $\#\mathcal{R}_{16,0,0} = \#\mathbf{R}$ (but what is $\mathcal{R}_{16,0,0} \cap \mathscr{D}$?)
- $\mathcal{R}_{16,0,0} \cap \overline{\mathbf{Q}(x)}$ are exactly the regular functions on some $Y_0(N) \cup \{i\infty\}$ expressed as power series in $x = \lambda/16$ (CDT 2021, Unbounded Denominators Conjecture)

Locus of (A, b, σ) for $\mathcal{R}_{A,b,\sigma} \cap \mathscr{D}$ to be finite-dimensional over $\mathbf{Q}(x)$?

An infinite-dimensional space

Proposition

The $\mathbf{Q}(x)$-linear span of the G-functions of geometric origin in $\mathcal{R}_{1,2,2}$ is infinite-dimensional.

Proof:

It contains the Hadamard product of the $x \rightsquigarrow 16x$ change of $\mathcal{R}_{16,0,0} \cap \overline{\mathbf{Q}(x)}$ with the G-function of geometric origin

$$g(x) := {}_3F_2\left[\begin{matrix} 1 & 1 & 1 \\ 1/2 & 1/2 \end{matrix}; \frac{x}{16}\right] = \sum_{n=0}^{\infty} \frac{x^n}{\binom{2n}{n}^2}.$$

The ODE singularities for Hadamard products multiply:

$$\{0, 1/16, \infty\} \times \{0, 16, \infty\} = \{0, 1, \infty\}.$$

The $\mathcal{O}(\widetilde{\mathcal{V}})$-module structure

The following is a $\mathbf{Z}[x, 1/(1-x), x \cdot (d/dx)]$-module:

$$\mathcal{D} := \mathcal{O}\left(\mathbf{C} \smallsetminus [1,\infty)\right) \cap \bigoplus_{n=0}^{\infty} \frac{x^n}{[1,\ldots,n]}\,\mathbf{Z}$$

$$= \mathrm{Span}_{\mathbf{Z}[x,1/(1-x)]}\left\{\log(1-x);\ 1;\ x^q/q : \text{prime powers } q\right\}.$$

This formula contains the irrationality statement $\log 2 \notin \mathbf{Q}$: otherwise

$$b\,\frac{\log(1-x) - \log 2}{1+x} \in \mathcal{D} \smallsetminus \mathrm{Span}(\cdots) \quad \text{if } \log 2 = a/b \in \mathbf{Q}.$$

Modified example: If we change the denominator type to $[1,\ldots,101n/100]$, we still have

$$\mathcal{D} \cap \mathbf{Z}[\![x]\!] = \mathcal{O}(\widetilde{\mathcal{V}}) = \mathbf{Z}[x, 1/(1-x)],$$

but $\mathcal{D} \otimes_{\mathbf{Z}} \mathbf{Q}$ is infinite over $\mathcal{O}(\widetilde{\mathcal{V}})_{\mathbf{Q}} = \mathbf{Q}[x, 1/(1-x)]$ and finite (free) module only over $\mathcal{O}(\widetilde{\mathcal{V}})_{\mathbf{Q}}[1/x] = \mathbf{Q}[x, 1/x, (1-x)]$

An example realizing $L(2,\chi_{-3}) = \sum_{n=0}^{\infty}\left(\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2}\right)$

$y := x^2/(x-1) = x \cdot x/(x-1) = x + x/(x-1)$, symmetrize

$$f(y) := \mathrm{Sym}^{-}(\mathrm{Li}_2)(y) := \left(x - \frac{x}{x-1}\right)\left(\mathrm{Li}_2(x) - \mathrm{Li}_2\left(\frac{x}{x-1}\right)\right)$$

$$\in \bigoplus_{n=0}^{\infty} \frac{y^n}{[1,\ldots,2n]^2}\mathbf{Z},$$

holonomic on $Y_0(2) \smallsetminus \{\text{elliptic point}\} = \mathbf{P}^1 \smallsetminus \{0,4,\infty\}$ (with $\mathbf{Z}/2$ local monodromy around the elliptic point $y=4$)

- turned holomorphic on $z \in \mathbf{D}$ under the substitution $y := h(z) = \lambda(\sqrt{z})^2/(\lambda(\sqrt{z})-1) = -256z + \cdots$
- $\varphi(z) := -\frac{64z(1+z)^2}{(1-z)^4}$ still has $|\varphi'(0)| = 64 > e^4$ and $f(\varphi(z)) \in \mathcal{O}(\mathbf{D})$, but now $\#\varphi^{-1}(1) = 1$
- $L(2,\chi_{-3}) = -\frac{2}{9}f(1)$

The André–Beukers theorem on E-functions

An *E-function* is a power series $\sum_{n=0}^{\infty} a_n x^n/n!$ such that $\sum_{n=0}^{\infty} a_n x^n$ is a G-function.

From [Beukers, *A refined version of the Siegel–Shidlovskii theorem* (2006)], using [André, *Séries Gevrey de type arithmétique*]:

Theorem (André, Beukers)

The ring of E-functions in $\mathbf{Q}[\![x]\!]$ generates over $\mathbf{Q}[x, 1/x]$ a free submodule of $\mathbf{Q}((x))$.

This statement contains and refines the full Siegel–Shidlovsky theorem on the relations among special values of E-functions:

$$f(x) \rightsquigarrow \frac{f(x) - f(1)}{1 - x} \qquad \text{if } f(1) \in \mathbf{Q}.$$

André's take on the Siegel–Shidlovsky theorem on E-functions

Fourier–Laplace transform: $x \mapsto -\frac{d}{dx}$, $\frac{d}{dx} \mapsto x$,
$F = \sum a_n \frac{x^n}{n!} \leftrightarrow f = \sum a_n x^n$, $x\frac{d}{dx}F \leftrightarrow x\frac{d}{dx}f$, $xF \leftrightarrow \left(x^2\frac{d}{dx} + x\right) f$

- *The Bezivin–Robba proof of Hermite–Lindemann–Weierstrass*
  The theorem easily reduces to prove that if $F(x)$ is an exponential polynomial over $\mathbf{Q}$ that vanishes at $F(1) = 0$, then $G(x) := F(x)/(x-1)$ is also an exponential polynomial. Fourier dual to an ODE with an irregular singularity at $x = 0$:
  If $\left(x^2\frac{d}{dx} + x - 1\right) g = f \in \mathbf{Q}(x)$ and g has a positive convergence radius at $x = 0$, then $g \in \mathbf{Q}(x)$.
- André's take: Chudnovsky's fundamental G-functions theorem to the order-2 linear differential system $\rightsquigarrow g \in \mathbf{Q}(x)$ (Honda–Katz since $x = 0$ is an irregular singular point here)

$$x^2\frac{d}{dx}\begin{pmatrix} g \\ 1 \end{pmatrix} = \begin{pmatrix} 1-x & f \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g \\ 1 \end{pmatrix}$$

Yves André's *Transcendance sans transcendance*

$$x \mapsto -\frac{d}{dx}, \qquad \frac{d}{dx} \mapsto x$$

The Fourier dual of "$x = 0$ is a regular (Fuchsian) singular point" is just "absence of singularities outside $\{0, \infty\}$"

5. Autour de Laplace. Preuve de 4.3 et 4.6

5.1. *Pentes.* Soit $\varphi = \sum_{i=0}^{\mu} \sum_{j=0}^{\nu} a_{i,j} z^j \frac{d^i}{dz^i} \in K[z, \frac{d}{dz}]$; son transformé de Fourier-Laplace est

$$\Finv\varphi = \sum_{j=0}^{\nu} \sum_{i=0}^{\mu} (-1)^j a_{i,j} \frac{d^j}{dz^j} z^i = \sum_{j=0}^{\nu} \sum_{i=0}^{\mu} b_{j,i} z^i \frac{d^j}{dz^j}.$$

Tous les opérateurs différentiels que nous considérons sont de type exponentiel au sens de [M, XII], c'est-à-dire vérifient $a_{\mu,\nu} \neq 0$ (ce qui équivaut à $b_{\nu,\mu} \neq 0$). En effet nous ne considérons que des opérateurs différentiels réguliers en l'infini, ou facteurs de transformés de Fourier de tels.

Notons d'autre part que si φ est régulier en l'infini, on a alors $a_{i\nu} = 0$ pour $i < \mu$, donc $b_{\nu,i} = 0$ pour $i < \mu$, ce qui signifie que $\Finv\varphi$ n'a de singularités qu'en 0 et ∞. Ceci établit i).

An analogous quest for G-functions?

Applying a *rational holonomy bound* (a $\mathbf{Q}(x)$-dimensional estimate) to the integrals $\int f(x)\,dx$ over $f(x) \in \mathcal{D}$, the $\mathcal{O}(\widetilde{\mathcal{V}})[1/h]$-finiteness of the integral holonomic module $\mathcal{D}$ is certainly true for some $h \in \mathcal{O}(\widetilde{\mathcal{V}}) \setminus \{0\}$.

Alas, this argument is completely ineffective, and does not allow for irrationality proofs for specific special values.

Is it true that one can always take $h = x$ in this statement?

The rational arithmetic holonomy bound

Theorem (CDT § 2.5 — after Bost–Charles)

For $Ae^{b\sigma} < 16$, we have the following bound more precisely on the finite $\mathbf{Q}(x)$-linear span of $\mathcal{R}_{A,b,\sigma}$. Choose any holomorphic map $\varphi : \mathbf{D} \to \mathbf{C} \smallsetminus \{1\}$ with $\varphi^{-1}(0) = \{0\}$ and $|\varphi'(0)| > Ae^{b\sigma}$. Then

$$\dim_{\mathbf{Q}(x)} \left(\mathcal{R}_{A,b,\sigma} \otimes_{\mathbf{Z}[x]} \mathbf{Q}(x)\right) \leq \frac{\iint_{\mathbf{T}^2} \log|\varphi(z) - \varphi(w)| \, \mu_{\mathrm{Haar}}(z)\mu_{\mathrm{Haar}}(w)}{\log|\varphi'(0)| - \log A - b\sigma}.$$

Example. For $\varphi : \mathbf{D} \hookrightarrow \mathbf{C}$ univalent, the double integral $= \log|\varphi'(0)|$.

The concluding slides that follow give essentially a complete proof — limiting for simplicity to the case $\varphi(z) = 4z/(1+z)^2$ — of the univalent case, in a slightly more precise form of the denominator term from exploiting that at least the function "1" has better denominators than the rest.

Proof of the characterization of the logarithm

We only need to prove that *if* $b\sigma < \log 4$, then:

$$m \leq \frac{\log 4}{\log 4 - b\sigma + b\sigma/m^2}$$

under the condition of a $\mathbf{Q}(x)$-linearly independent m-tuple $f_1(x), \ldots, f_m(x) \in \mathbf{Q}[\![x]\!]$ of type $[1, \ldots, bn]^\sigma$ and holomorphic on $\mathbf{C} \setminus [1, \infty)$.

Indeed,

$$\frac{\log 4}{\log 4 - 8/9} = 2.78705\ldots < 3$$

then shows that there is no third function of type $[1, \ldots n]$ besides the two known examples $f_1 = 1$ and $f_2 = \log(1-x)$.

Example: The case $m = 2$ *is exactly Zudilin's criterion.*

Proof of the characterization of the logarithm

WTS: $m \leq \frac{\log 4}{\log 4 - b\sigma + b\sigma/m^2}$
WLOG, $f_1 \equiv 1$.

Let $q_n(x) := 4^{-n} \cdot 2T_n(2x-1) = 4^{-n} \cdot 2\cos(n \arccos(2x-1)) = x^n + \cdots$ be the normalized Chebyshev polynomials. They are monic and have $\sup_{[0,1]} |q_n| = 2 \cdot 4^{-n}$.

Evaluation module. The free **Z**-module of rank mD:

$$E_D := \oplus_{n=0}^{D-1} \frac{x^{D-n}}{[1,\ldots,bn]^{\sigma}} \cdot f_1(x) \oplus \bigoplus_{j=2}^{m} \mathbf{Z}[x]_{<D} \cdot f_j(x);$$

$$\mathbf{Z}[x]_{<D} \ni Q(x) = \sum_{n=0}^{D-1} c_n(Q) \cdot q_n(x), \quad |c_n(Q)| < T \cdot 4^n$$

$\rightsquigarrow$ lattice point count with $\sim T^D 4^{\binom{D}{2}+o(D^2)}$ solutions

$\rightsquigarrow \mathcal{I}_D(T) \subset E_D$ of $\#\mathcal{I}_D(T) \sim T^{mD} 4^{m\binom{D}{2}+o(D^2)} \cdot e^{\sigma b D^2/2}$.

Proof of the characterization of the logarithm

WTS: $m \leq \frac{\log 4}{\log 4 - b\sigma + b\sigma/m^2}$

Evaluation module.

$$E_D \hookrightarrow \mathbf{Q}[\![x]\!], \quad (Q_1, \ldots, Q_m) \mapsto F(x) := \sum_{i=1}^{m} x^D Q_i(1/x) f_i(x),$$
$$= cx^n + \ldots, \quad c \in \frac{1}{[1, \ldots, bn]^\sigma} \mathbf{Z} \smallsetminus \{0\}.$$

Vanishing filtration jumps. There are exactly mD possible values $u(1) < \cdots < u(mD)$ for n.

Interval of coefficient possibilities. If $n = u(j)$ is one of those vanishing order values for the difference $V = F_1 - F_2$ of two elements $F_1, F_2 \in \mathcal{I}_D(T)$, then

$$c = \frac{1}{2\pi i} \oint_{\Gamma \supset [0,1]} V(1/z) q_n(z) \, \frac{dz}{z} = T 4^{-n + o(n+D)}.$$

Proof of the characterization of the logarithm

Upper bound on $\#\mathcal{I}_D(T)$*.* There results (Perelli and Zannier's idea) the upper bound:

$$\#\mathcal{I}_D(T) \leq \prod_{j=1}^{mD} \left(1 + T \cdot [1, \ldots, bu(j)]^{\sigma} \cdot 4^{-u(j)+o(u(j)+D)}\right).$$

Asymptotics $T \to \infty, D \to \infty$*.* But recall $\#\mathcal{I}_D(T) \sim T^{mD} 4^{m\binom{D}{2}+o(D^2)} \cdot e^{\sigma b D^2/2}$. The comparison filters into the requisite bound

$$\begin{aligned}\binom{mD}{2}(\log 4 - b\sigma) &\leq (\log 4 - b\sigma)\left(\sum_{j=1}^{mD} u(j)\right) \\ &\leq \log 4 \cdot mD^2/2 + \sigma b \cdot D^2/2 \\ \rightsquigarrow m &\leq \frac{\log 4}{\log 4 - b\sigma + b\sigma/m^2}.\end{aligned}$$

Thank you for your attention!

Unused slides

Application: A mixed periods example

Consider now with the bivalent map

$$\varphi(z) := \frac{8(z+z^3)}{(1+z)^4} = 1 - \left(\frac{1-z}{1+z}\right)^4, \quad \mathbf{D} \to \mathbf{C} \setminus \{1\}, \quad \varphi^{-1}(0) = \{0\}$$

(the next Landen iterative layer), which takes either connected component of $\mathbf{D} \setminus (-1,1)$ conformally isomorphically onto $\mathbf{C} \setminus (-\infty, 1]$, but bijects

$$\varphi^{-1}(-\infty, 1] \longleftrightarrow (-1,1).$$

Application: A mixed periods example

$\varphi(z) := \frac{8(z+z^3)}{(1+z)^4}$: bivalent, but still simple enough to have its Bost–Charles integral computed exactly.

Essentially following Smyth's calculations of the bivariate Mahler measure $m(1+x+y-xy) = 2G/\pi$, where $G = L(2, \chi_{-4})$ is the Catalan constant:

Lemma

$$\iint_{\mathbf{T}^2} \log|\varphi(z) - \varphi(w)| \, \mu_{\mathrm{Haar}}(z)\mu_{\mathrm{Haar}}(w) = \log 8 + \frac{4G}{\pi}$$

$$\frac{\varphi(z)-\varphi(w)}{z-w} = 8\frac{(1-zw)(1+ix-iy-xy)(1-ix+iy-xy)}{(1+z)^4(1+w)^4}.$$

Application: A mixed periods example

This choices gives us completely similarly the "δ is a nonapparent singularity" case of:

Theorem

Suppose $f(x) \in \bigoplus_{n=0}^{\infty} \frac{x^n}{[1,\ldots,n][1,\ldots,n/2]} \mathbf{Z}$ *is holomorphic in* $\mathbf{C} \setminus [1,\infty)$*, and is analytically continuable as a holomorphic function along all paths in* $\mathbb{P}^1 \setminus \{0,\delta,1,\infty\}$*, for some* $\delta \in (-\infty,-1)$*.*
Then,

$$f(x) = Q_0(x) + Q_1(x)\log(1-x) + Q_2(x)\log^2(1-x)$$

for some rational functions $Q_0, Q_1, Q_2 \in \left[x, \frac{1}{x}, \frac{1}{1-x}\right] \subset \mathbf{Q}(x)$*.*

Observe that $f\left(\frac{8(z+z^3)}{(1+z)^4}\right) \in \mathcal{O}(\mathbf{D})$*.*

Application: A mixed periods example

Proof of the "non-$\mathbf{P}^1 \setminus \{0,1,\infty\}$ case":

$$1, \quad \log(1-x), \quad \log^2(1-x), \quad f(x), \quad f\left(\frac{x}{x-1}\right)$$

$$\rightsquigarrow \mathbf{b} := \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}^t, \qquad \tau(\mathbf{b}) = \frac{69}{50} = 1.38.$$

The rational holonomy bounds reads

$$5 = m \leq \frac{\log 8 + \frac{4G}{\pi}}{\log 8 - 69/50} = 4.640395\ldots,$$

a contradiction.

Application: A mixed periods example

$$H_A(x) := \frac{1}{\sqrt{1-4x}} \in \mathbf{Z}[\![x]\!],$$
$$H_B(x) := \frac{1}{\sqrt{1-4x}} \int_0^x \frac{1}{1-t} \frac{1}{\sqrt{1-4t}}\, dt \in \mathbf{Q}[\![x]\!],$$
$$H_C(x) := \frac{1}{\sqrt{1-4x}} \int_0^x \frac{\log(1-t)}{t\sqrt{1-4t}}\, dt \in \mathbf{Q}[\![x]\!],$$
$$H_D(x) := \frac{1}{\sqrt{1-4x}} \int_0^x \frac{\log(1-t)}{1-t} \frac{1}{\sqrt{1-4t}}\, dt \in \mathbf{Q}[\![x]\!].$$

But the $x = 1/4$ local monodromy operator is lower-triangular with period matrix $\rightsquigarrow$ $\mathbf{Q}$-*linear independence proof for the first column*

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ -2L(1,\chi_{-3}) & 1 & 0 & 0 \\ \frac{\pi^2}{9} & 0 & 1 & 0 \\ -2\left(L(1,\chi_{-3})\log 3 - L(2,\chi_{-3})\right) & 0 & 0 & 1 \end{pmatrix}$$

Application: Irrationality proof for a Mahler measure

Corollary

$$m\left(\frac{(1+x+y)^4}{3}\right) = \frac{3L(2,\chi_{-2}) - \frac{\pi}{\sqrt{3}}\log 3}{\pi/\sqrt{3}} \notin \mathbf{Q}.$$

The classical Hermite–Padé approximants to $\log(1-x)$

$$2\sum_{j=0}^{k}\binom{k}{j}^2 (H_{n-k}-H_j)(1-y)^j + \log(1-y)\sum_{j=0}^{k}\binom{k}{j}^2(1-y)^j$$

$$= y^{2k+1}\int_0^1 \frac{t^k(t-1)^k}{(ty-1)^{k+1}}\,dt = -\frac{y^{2k+1}}{(2k+1)\binom{2k}{k}} + O(y^{2k+2}),$$

- *Set* $y := 1/n$ *and take the* x^k *generating function: holonomic*$^{+}$ *on* $\mathbf{C} \setminus \left\{ \left(\frac{1\pm\sqrt{1-1/n}}{1/n} \right)^2 \right\}$
- *Set* $y := 1/m$ *and take the* x^k *generating function: holonomic*$^{+}$ *on* $\mathbf{C} \setminus \left\{ \left(\frac{1\pm\sqrt{1-1/m}}{1/m} \right)^2 \right\}$
- *Form the Hadamard product construction*

Application: irrationality of $\log(1-1/n)\log(1-1/m)$ for $|1-m/n|<10^{-6}$

$$\sum_{k=0}^{\infty}(b_k - a_k\mathrm{Li}_1(1/n))\,x^k, \quad \sum_{k=0}^{\infty}(v_k - u_k\mathrm{Li}_1(1/m))\,x^k$$

$$\rightsquigarrow \sum_{k=0}^{\infty}(b_k v_k - a_k u_k\mathrm{Li}_1(1/n)\mathrm{Li}_1(1/m))\,(nmx)^k$$

- the type would be $[1,\ldots,k]^2$
- the ODE singularities would be (say, $m,n>0$):

$$0; \quad nm\left(1-\sqrt{1-1/n}\right)^2\left(1-\sqrt{1-1/m}\right)^2 \text{ (overconvergent!)};$$

$$nm\left(1\pm\sqrt{1-1/n}\right)^2\left(1\mp\sqrt{1-1/m}\right)^2 = 1+o_{|n/m|\to 1}(1);$$

$$nm\left(1+\sqrt{1-1/n}\right)^2\left(1+\sqrt{1-1/m}\right)^2 \to \infty$$

Prévost's interpretation of Apéry's overconvergent function

Is there any simultaneous Hermite–Padé interpretation of the function $H(x)$ in analogy to:

$[2k/2k]$ Hermite–Padé approximants to $\zeta(3, 1 + 1/y)$ give:

$$\left\{ \left(\sum_{k=0}^{\infty} \binom{n}{k} \binom{n+k}{k} \binom{1/y}{k} \binom{k+1/y}{k} \right) \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{1/y}{m} \binom{m+1/y}{m}} - \left(\sum_{k=1}^{\infty} \frac{1}{(k+1/y)^3} \right) \left(\sum_{k=0}^{\infty} \binom{n}{k} \binom{n+k}{k} \binom{1/y}{k} \binom{k+1/y}{k} \right) \right\} y^{2n} = O(y^{4n+1})$$

Setting $y := 1/n$, then the x^n generating function recovers the *Apéry overconvergent function*

$$G(x) = B(x) - \zeta(3) A(x), \qquad \text{type } [1, \ldots, n]^3$$

This writes down an $\{0, (\sqrt{2}-1)^4\}$-overconvergent branch of a holonomic function on $\mathbf{C} \setminus \{0, (\sqrt{2} \pm 1)^4\}$. *(which lives more naturally on $X_1(6)^+$)

Setting up an approach for the $\mathbf{Q}$-linear independence of $1, \zeta(2)$, and $L(2, \chi_{-3})$

Conjecture

The $\mathbf{Q}\left[x, \frac{1}{x}, \frac{1}{1-x}\right]$*-linear span of the five G-functions*

$$1, \quad \mathrm{Li}_1(x) = \log(1-x), \quad \mathrm{Li}_{1,1}(x) = \log^2(1-x),$$
$$\mathrm{Li}_2(x), \qquad \frac{1}{\sqrt{1-x}} \int_0^x \frac{\log(1-t)}{t\sqrt{1-t}}\, dt$$

ought probably to be characterized arithmetically as:

$\exists T \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$ *(finite) such that*

$$f(x) \in \mathbf{Q}[\![x]\!] \cap \mathcal{O}\left(\mathbf{C} \setminus [1, \infty)\right)$$

has type $[1, \ldots, n]^2$ *and holomorphic analytic continuation along all paths in* $\mathbf{CP}^1 \setminus \{0, 1, \infty\} \cup T$

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Integral Solutions of Apéry-like Recurrence Equations

Don Zagier

ABSTRACT. In [4], Beukers studies the differential equation

$$(*)\qquad \big((t^3+at^2+bt)F'(t)\big)' + (t-\lambda)F(t) = 0,$$

where a, b and λ are rational parameters, and asks for which values of these parameters this equation has a solution in $\mathbb{Z}[\![t]\!]$, the motivating example being the Apéry sequence with $a=11$, $b=-1$, $\lambda=-3$. We describe a search over a suitably chosen domain of 100 million triples (a,b,λ). In this domain there are 36 triples yielding integral solutions of (*). These can be further subdivided into members of four infinite classes, two of which are degenerate special cases of the other two, and seven sporadic solutions. Of these solutions, twelve, including all the sporadic ones, have parametrizations of Beukers type in terms of modular forms and functions. These solutions are related to elliptic curves over $\mathbb{P}^1$ with four singular fibres.

Case	β	α	C	c_1	c_2	L
A	8	-1	$\dfrac{2}{\pi\sqrt{3}}$	$\dfrac{1}{3}$	$\dfrac{2}{3}$	$\dfrac{1}{4}\zeta(2) = 0.4112335\ldots$
C	9	1	$\dfrac{3\sqrt{3}}{4\pi}$	$\dfrac{1}{4}$	$\dfrac{3}{4}$	$\dfrac{1}{2}L_{-3}(2) = 0.3906512\ldots$
D	ϕ^5	$-\phi^{-5}$	$\dfrac{\phi^{5/2}}{2\pi\sqrt[4]{5}}$	$\dfrac{1}{\phi\sqrt{5}}$	$\dfrac{\phi}{\sqrt{5}}$	$\dfrac{1}{5}\zeta(2) = 0.3289868\ldots$
E	8	4	$\dfrac{2}{\pi}$	0	1	$\dfrac{1}{2}L_{-4}(2) = 0.4579827\ldots$
F	9	8	$\dfrac{3\sqrt{3}}{\pi}$	-2	3	$\dfrac{5}{8}L_{-3}(2) = 0.4883140\ldots$

In each of the five cases of Table 5, formula (10) gives an interesting series of rational approximations for one of the numbers $\zeta(2)$, $L_{-3}(2)$ or $L_{-4}(2)$, but, as already mentioned, except in case **D** these do not converge quickly enough to yield the irrationality of the limit. Indeed, from the table we see that in cases **E** and **F** the quantity $v_n - Lu_n$ blows up exponentially like $4^n/n$ or $8^n/n$, respectively, and even in cases **A** and **C**, where $|\alpha| = 1$ and hence $v_n - Lu_n$ tends to zero like $O(1/n)$, this is not enough to give the irrationality of L because v_n itself has a denominator which blows up like e^{2n}. The speed of convergence is best in these two cases, with $v_n/u_n - L$ being of the order of $\left(\frac{1}{8}\right)^n$ and $\left(\frac{1}{9}\right)^n$, respectively, while the convergence in cases **E** (Catalan's constant) and **F** is only like $\left(\frac{1}{2}\right)^n$ and $\left(\frac{8}{9}\right)^n$, respectively. The approximations lead in each of the five cases to a simple infinite continued fraction

$L(2,\chi_{-3})$ and $\zeta(2)$ as periods of Eisenstein series in $M_3(\Gamma_0(6),\chi_{-3})$

Take these Eichler integrals of Eisenstein series:

$$B := \sum_{n=1}^{\infty}\left(\sum_{d|n}(-1)^{d-1}\chi_{-3}(n/d)d^2\right)\frac{q^n}{n^2},$$

$$C := \sum_{n=1}^{\infty}\left(\sum_{d|n}\chi_{-3}(d)d^2\right)\frac{q^n}{n^2} - \sum_{n=1}^{\infty}\left(\sum_{d|n}\chi_{-3}(d)d^2\right)\frac{q^{2n}}{(2n)^2}$$

We did $f := \sum a_n q^n \rightsquigarrow \sum \frac{a_n}{n^{k-1}} q^n =: F$ starting from weight $k = 3$ because of Bol's identity:

$$\left(q\frac{d}{dq}\right)^{k-1}\left(F\big|_{2-k}\gamma\right) = \left(\frac{d^{k-1}}{d\tau^{k-1}}F\right)\Big|_k\gamma = f|_k\gamma = f = \left(q\frac{d}{dq}\right)^{k-1}(F)$$

$$\rightsquigarrow F\big|_{2-k}\gamma = F + [\text{period polynomial in } \tau \text{ of degree} \leq k-2]$$

$L(2, \chi_{-3})/2$ is the w_6-period of B

Choose $\gamma = w_6$*, the involution exchanging the cusps* $\tau = i\infty$ *and* $\tau = 0$ *of* $\Gamma_0(6)$*:*

$$\left(B(q) - \lim_{q \to 1} B(q) \right) \Big|_{-1} w_6 = B(q) - \lim_{q \to 1} B(q)$$

This *Apéry limit* is:

$$\lim_{q \to 1} B(q) = \lim_{q \to 1} \left\{ \sum_{n=1}^{\infty} \chi_{-3}(n) n^{-2} \frac{q^n}{1 + q^n} \right\} = \frac{1}{2} L(2, \chi_{-3})$$

$\zeta(2)/4$ is the w_6-period of C

Choose $\gamma = w_6$*, the involution exchanging the cusps* $\tau = i\infty$ *and* $\tau = 0$ *of* $\Gamma_0(6)$*:*

$$\left(C(q) - \lim_{q \to 1} C(q) \right) \Big|_{-1} w_6 = C(q) - \lim_{q \to 1} C(q)$$

This *Apéry limit* is:

$$\lim_{q \to 1} C(q) = \lim_{q \to 1} \left\{ \frac{1}{4} \sum_{n=1}^{\infty} \chi_{-3}(n) \left(4\mathrm{Li}_2(q^n) - \mathrm{Li}_2(q^{2n}) \right) \right\} = \frac{1}{4}\zeta(2)$$

Simultaneous approximating forms to the two Eichler periods $L(2,\chi_{-3})/2$ and $\zeta(2)/4$

$$\left(B - \frac{1}{2}L(2,\chi_{-3})\right)\Big|_{-1} w_6 = B - \frac{1}{2}L(2,\chi_{-3}),$$
$$\left(C - \frac{1}{4}\zeta(2)\right)\Big|_{-1} w_6 = C - \frac{1}{4}\zeta(2)$$

Kill the automorphy factor by multiplying by a modular form of the opposite weight $+1$.

We follow Zagier's choice:

$$\theta_{-3}(\tau) := \sum_{m,n\in\mathbf{Z}} q^{m^2+mn+n^2} \in M_1(\Gamma_0(3),\chi_{-3})$$
$$A := \frac{\theta_{-3}(\tau)+\theta_{-3}(2\tau)}{2} \in M_1(\Gamma_0(6),\chi_{-3})$$

Simultaneous approximating forms to the two Eichler periods $L(2,\chi_{-3})/2$ and $\zeta(2)/4$

Now both $A\cdot\left(B-\frac{1}{2}L(2,\chi_{-3})\right)$ and $A\cdot\left(C-\frac{1}{4}\zeta(2)\right)$ are regular in either of the cusps $\tau=i\infty$ and $\tau=0$ of $Y_0(6)$
Apéry's key: $\mathbf{Z}[\![q]\!]=\mathbf{Z}[\![x]\!]$, turning modular forms into G-functions:

$$x=q\prod_{n=1}^{\infty}\frac{(1-q^n)^4(1-q^{6n})^8}{(1-q^{2n})^8(1-q^{3n})^4}=q-4q^2+10q^3+\dots$$

$$X_0(6)\smallsetminus\{i\infty,0,1/3,1/2\}=Y_0(6)=\mathbf{H}/\Gamma_0(6)\overset{\cong}{\to}\mathbf{P}^1\smallsetminus\{0,1/9,1,\infty\}$$

$$A=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\right)x^n\in\mathbf{Z}[\![x]\!]\cap\mathrm{Hol}\left(\mathbf{P}^1\smallsetminus\left\{0,\frac{1}{9},1,\infty\right\}\right)$$

The Picard–Fuchs equation

Explicitly:

$$L(A) = 0, \quad L(AB) = 1, \qquad L(AC) = \frac{1}{1-x},$$

where

$$L := x(1-x)(1-9x)\frac{d^2}{dx^2} + (1 - 20x + 27x^2)\frac{d}{dx} + (9x - 3)$$

Beukers integral:

$$H = \frac{1}{2}L(2,\chi_{-3})A - AB$$

$$= \sum_{n=0}^{\infty} x^n \iint_{[0,1]^2} \frac{9^n s^n t^n (1-s^3)^n (1-t^3)^n}{(1+st+s^2t^2)^{2n+1}}\, ds\, dt$$

Type $[1,\ldots,n]^2$ $\qquad H(\lambda(z/2)) \in \mathcal{O}(\mathbf{D}),\ |\varphi'(0)| = 8 > e^2$

The Apéry limits method

The holonomic construction

$AB - \frac{1}{2}L(2,\chi_{-3})A$ and $AC - \frac{1}{4}\zeta(2)A$ are regular not only at the singularity $x = 0$, but also at the next singularity $x = 1/9$.

$\rightsquigarrow$ so are all their $\mathbf{C}$-linear combinations

$\rightsquigarrow$ a $\mathbf{Q}$-linear dependency of the periods

$\Rightarrow H \in \mathbf{Q}[\![x]\!]$ with the same overconvergence properties